TEMPERATURE DISTRIBUTION IN A PLANAR BODY WITH CIRCULAR BOUNDARY AND THERMALLY INSULATED CRACKS

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UDC 536.12

Stationary heat-conduction problems are analyzed for infinite plates with a circular hole or for a circular disk weakened by straight-line thermally insulated cracks of arbitrary direction if boundary conditions of the first or second kind are specified on the domain boundary.

Let N thermally insulated cuts (cracks) of length $2l_n$ ($n = 1, 2, \ldots, N$) be given in an elastic isotropic plane related to a Cartesian coordinate system xOy. The centers O_n of the cuts are given by the coordinates $z_n^0 = x_n^0 + iy_n^0$. The centers of the local coordinate systems $x_nO_ny_n$ whose axes O_nx_n are identical with those of the crack lines forming the angles α_n with the Ox axis are positioned at the points O_n.

It is assumed that in a body without cuts the temperature distribution is described by a given harmonic function $t_0(x, y)$. The general temperature field in a domain with cracks can be represented by the sum

$$T(x, y) = t_0(x, y) + t(x, y),$$

where t(x, y) is the perturbed temperature field due to the presence of cuts. The cracks are thermally insulated; therefore, on their boundaries the following conditions are satisfied:

$$\frac{\partial t^+(x, y)}{\partial y_n} = \frac{\partial t^-(x, y)}{\partial y_n} = -\left.\frac{\partial t_0(x, y)}{\partial y_n}\right|_{y_n=0} = q_n(x_n), \quad |x_n| < l_n. \tag{1}$$

The temperature field for an infinite plane weakened by a system of thermally insulated cracks can be represented in the form [1]

$$t_1(x, y) = \mathrm{Re}\,\frac{1}{\pi i}\sum_{k=1}^{N}\exp(i\alpha_k)\int_{-l_k}^{l_k}\frac{\gamma_k(t)\,dt}{T_k - z}, \quad T_k = t\exp(i\alpha_k) + z_k^0, \tag{2}$$
$$z = x + iy,$$

where $2\gamma_k(x)$ is the temperature jump due to the crossing of a crack line.

A similar representation of the temperature field will be found in an infinite plate with a circular hole of unit radius and center at the origin with N thermally insulated cracks. The boundary conditions are given of the first or the second kind at the boundary of the hole. It can be assumed without loss of generality that the contour γ is either thermally insulated or a zero temperature is maintained on it. Then the temperature field t(x, y) can be represented as a superposition of the solutions of two problems: The determination of the temperature (of the heat flux) on the hole boundary with the aid of the formula (2) if all cracks are located outside this unit circle $\gamma(|z| > 1,\ |T_k| > 1)$, and the determination of the solution of the heat-conduction problem for a plate with a circular hole and with a specified temperature (with heat flux) on its boundary which is equal in magnitude but has a sign opposite to the one found above.

By using the formula (2), one obtains on the contour γ $[z = \rho\exp(i\theta) = \rho\sigma]$ the following result:

$$t_1(x, y)|_\gamma = f(\sigma) = \frac{1}{2\pi i}\sum_{k=1}^{N}\int_{-l_k}^{l_k}\left[\frac{\exp(i\alpha_k)}{T_k - \sigma} + \frac{\sigma\exp(-i\alpha_k)}{1 - \sigma\bar{T}_k}\right]\gamma_k(t)\,dt. \tag{3}$$

Institute of Mechanics and Physics, Academy of Sciences of the Ukrainian SSSR, L'vov. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 32, No. 2, pp. 350-353, February, 1977. Original article submitted January 5, 1976.

The temperature field $t_2(x, y)$ in the plate with a hole under the condition that

$$t_2(x, y)|_\gamma = -f(\sigma) \tag{4}$$

is found from the relation [2]

$$t_2(x, y) = \operatorname{Re}\left[F_0(z) - \overline{F}_0\left(\frac{1}{z}\right)\right], \tag{5}$$

where

$$F_0(z) = \frac{1}{2\pi i}\int\limits_\gamma \frac{f(\sigma)\,d\sigma}{\sigma - z}. \tag{6}$$

Having evaluated the Cauchy integrals over the circumference γ of unit radius [3], from the formulas (3), (5), and (6) one obtains

$$t_2(x, y) = -\operatorname{Re}\frac{1}{\pi i}\sum_{k=1}^{N}\int\limits_{-l_k}^{l_k}\frac{z\exp(-i\alpha_k)}{1 - z\overline{T}_k}\gamma_k(t)\,dt. \tag{7}$$

By adding up the expressions (2) and (7), the perturbed temperature field t(x, y) is obtained in the case of an infinite plane with a circular hole which is stress free and on whose circumference zero temperature is specified, as well as a system of cracks with known temperature jumps over their lines.

Assuming that the cuts do not intersect, that they are all in the interior of the domain, that the boundary conditions (1) are satisfied on the boundaries of the cracks, and also that

$$\int\limits_{-l_k}^{l_k}\gamma_k'(t)\,dt = 0 \quad (k = 1, 2, \ldots, N),$$

one obtains a system of N singular integral equations of the heat-conduction problem for determining the unknown function $\gamma_n'(x)$:

$$\int\limits_{-l_k}^{l_n}\frac{\gamma_n'(t)\,dt}{t - x} + \sum_{k=1}^{N}\int\limits_{-l_k}^{l_k}\gamma_k'(t)P_{nk}(t, x)\,dt = \pi q_n(x), \qquad |x| < l_n \; (n = 1, 2, \ldots, N), \tag{8}$$

where

$$P_{nk}(t, x) = \operatorname{Re}\left\{\exp(i\alpha_n)\left[\frac{1 - \delta_{nk}}{T_k - X_n} + \frac{1}{X_n(1 - \overline{T}_k X_n)}\right]\right\}, \qquad X_n = x\exp(i\alpha_n) + z_n^0, \tag{9}$$

and δ_{nk} is the Kronecker's symbol.

Let us now consider the case of a thermally insulated boundary of a circular hole on a plane with cracks. Starting with the formula (2), one finds the value of the heat flux on the circumference γ:

$$\left.\frac{\partial t_1}{\partial \rho}\right|_\gamma = g(\sigma) = \frac{1}{2\pi i}\sum_{k=1}^{N}\int\limits_{-l_k}^{l_k}\left[\frac{\exp(i\alpha_k)}{(T_k - \sigma)^2} - \frac{\exp(-i\alpha_k)}{(1 - \sigma\overline{T}_k)^2}\right]\sigma\gamma_k(t)\,dt. \tag{10}$$

When the condition

$$\left.\frac{\partial t_2}{\partial \rho}\right|_\gamma = -g(\sigma) \tag{11}$$

is satisfied on the boundary γ, the temperature $t_2(x, y)$ in a plane with a hole is found by using the formula [2]

$$t_2(x, y) = \operatorname{Re}\left[F_0(z) + \overline{F}_0\left(\frac{1}{z}\right)\right]. \tag{12}$$

Here

$$F_0(z) = -\frac{1}{2\pi i}\int_{\gamma} g(\sigma)\ln(\sigma - z)\frac{d\sigma}{\sigma}. \tag{13}$$

Using the relations (12) together with the expressions (13) and (10), one obtains

$$t_2(x, y) = \mathrm{Re}\frac{1}{\pi i}\sum_{k=1}^{N}\int_{-l_k}^{l_k}\frac{\exp(-i\alpha_k)}{\overline{T}_k(1 - z\overline{T}_k)}\gamma_k(t)\,dt. \tag{14}$$

The sum of the expressions (2) and (14) provides the sought representation of the perturbed temperature field $t_2(x, y)$ for a plane with thermally insulated holes and cracks. The functions $\gamma_k'(x)$ are determined by the following system of N integral equations:

$$\int_{-l_n}^{l_n}\frac{\gamma_n'(t)\,dt}{t - x} + \sum_{k=1}^{N}\int_{-l_k}^{l_k}\gamma_k'(t)\,Q_{nk}(t, x)\,dt = \pi q_n(x), \tag{15}$$

$$|x| < l_n \ (n = 1, 2, \ldots, N).$$

The kernels $Q_{nk}(t, x)$ are given by

$$Q_{nk}(t, x) = \mathrm{Re}\left\{\exp(i\alpha_n)\left[\frac{1 - \delta_{nk}}{T_k - X_n} - \frac{1}{X_n(1 - \overline{T}_k X_n)}\right]\right\}. \tag{16}$$

It is noted that integral equations of the corresponding problems in heat conduction for a circular disk of unit radius and with a system of N thermally insulated cracks can be obtained in a similar manner. They are also of the form (8), (9), (15), and (16), the only difference being that in this case $|T_k| < 1$, $|X_n| < 1$.

The integral equations arrived at for any location of the cracks can be solved numerically [4]. If there are great distances between the cuts and between the cuts and the domain boundary, one can find analytic solutions of these equations [5] by using perturbation methods.

NOTATION

T, t_0, t, temperature; $2l$, $2l_n$, crack length; N, number of cracks; x, y, rectangular Cartesian coordinates; $2\gamma_k(x)$, temperature jump for crossing crack line.

LITERATURE CITED

1. M. P. Savruk and A. P. Datsyshin (O. P. Datsishin), in: Reports of the L'vov Polytechnic Institute No. 87, Mathematics and Mechanics [in Ukrainian], Vishcha Shkola, L'vov (1974).
2. I. A. Prusov, Some Problems of Thermoelasticity [in Russian], Izd. BGU im. V. I. Lenin, Minsk (1972).
3. G. N. Savin, Stress Distribution near Holes [in Russian], Naukova Dumka, Kiev (1968).
4. F. E. Erdogan and G. D. Gupta, Quart. Appl. Math., 29, No. 4 (1972).
5. M. P. Savruk and A. P. Datsyshin, Prikl. Mat. Mekh., 37, No. 2 (1973).